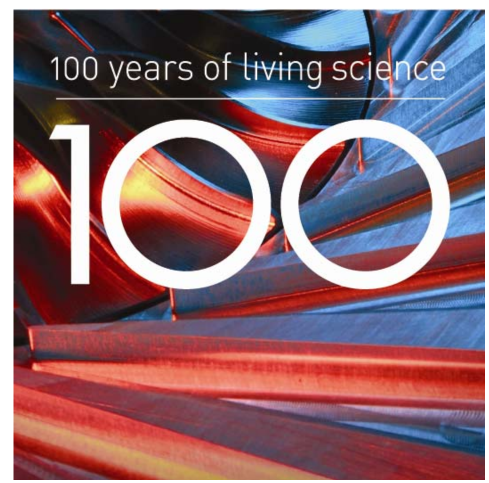
Oxford, 8th May 2007

Tim Evans
Theoretical Physics

Exact Results for Cultural Transmission and Network Rewiring

T.S.Evans, **A.D.K.Plato**

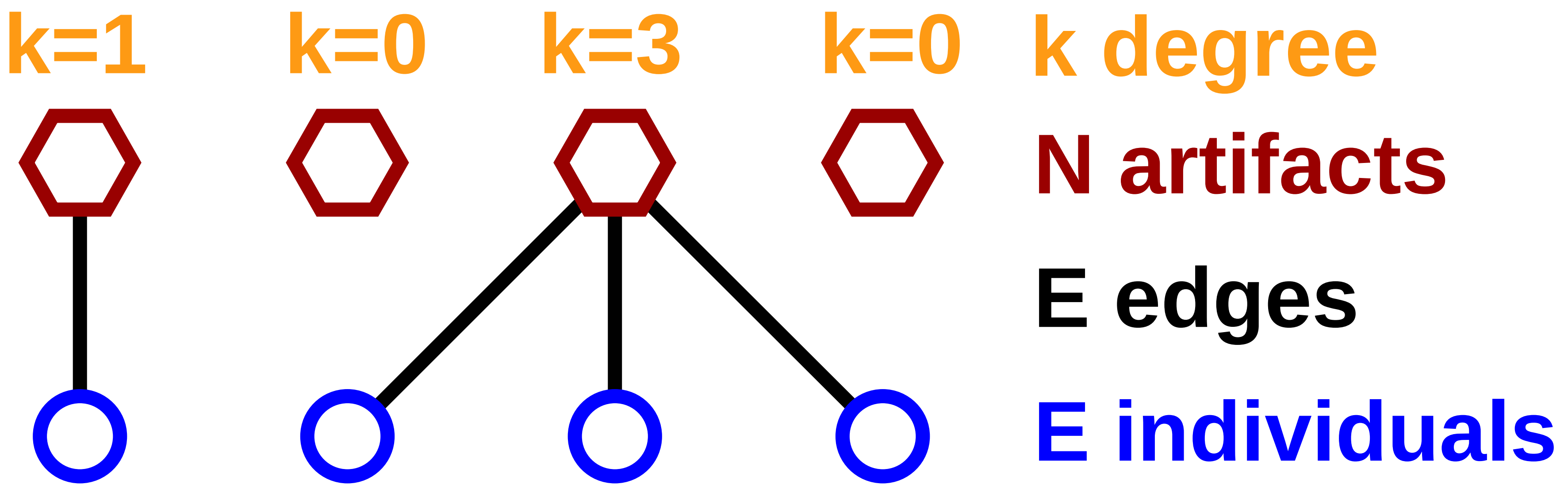
- *"Exact Solution for the Time Evolution of Network Rewiring Models"*
  Phys. Rev. E **75** (2007) 056101 [cond-mat/0612214]
- *"Network Rewiring Models"* (for ECCS07)
  www.imperial.ac.uk/people/t.evans



The Model

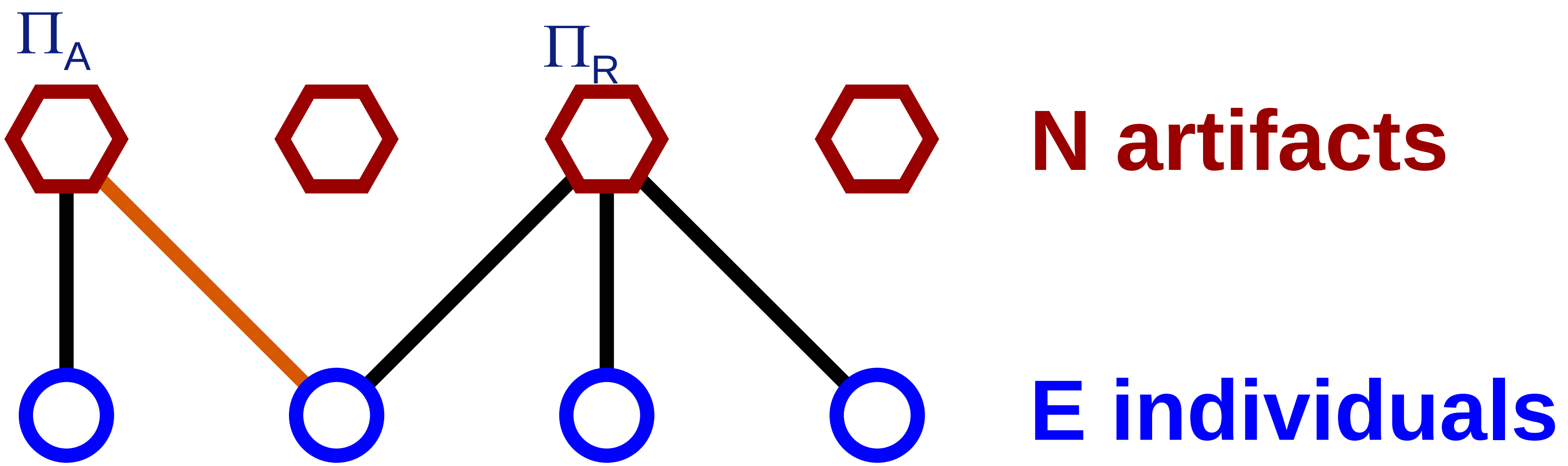
- Bipartite network
  E individual vertices each with one edge
  connected to N individual vertices

- Study degree k of artifact vertices
  $n(k)$ = degree distribution,
  $p(k) = n(k)/N$ = degree probability distribution



The Model - Rewiring

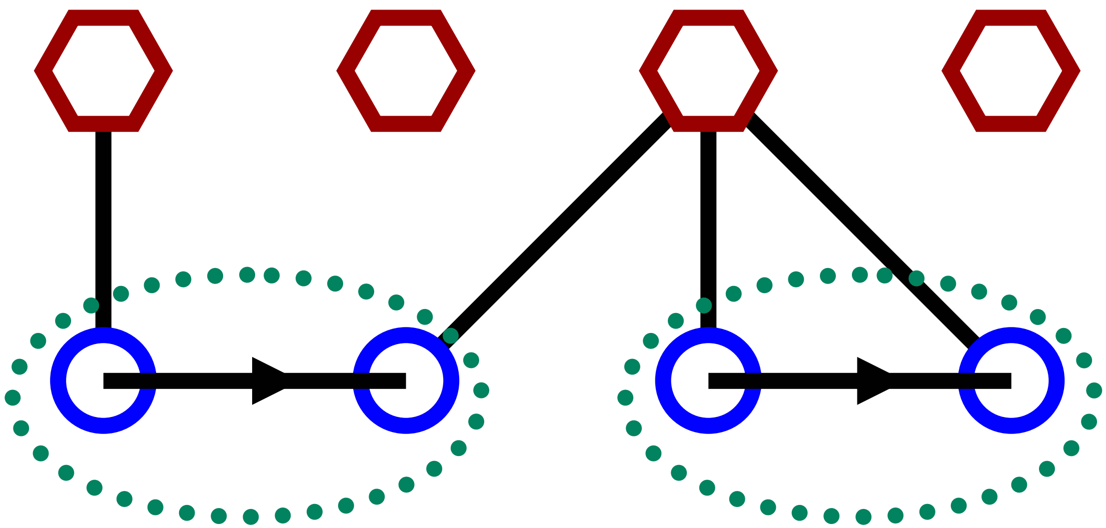
- **Removal**: Choose an edge intending to rewire its artifact end = choosing departure artifact with probability Π_R.
- **Attachment**: Choose an arrival artifact with probability Π_A ready to accept edge.
- **Rewire**: Only *after* these choices are made.



Equivalence to other network rewiring models

- **Directed/Undirected Network**:
  Join edges of individual vertices (2i) and (2i+1).

[Watts and Strogatz, 1998]



N artifacts

E edges

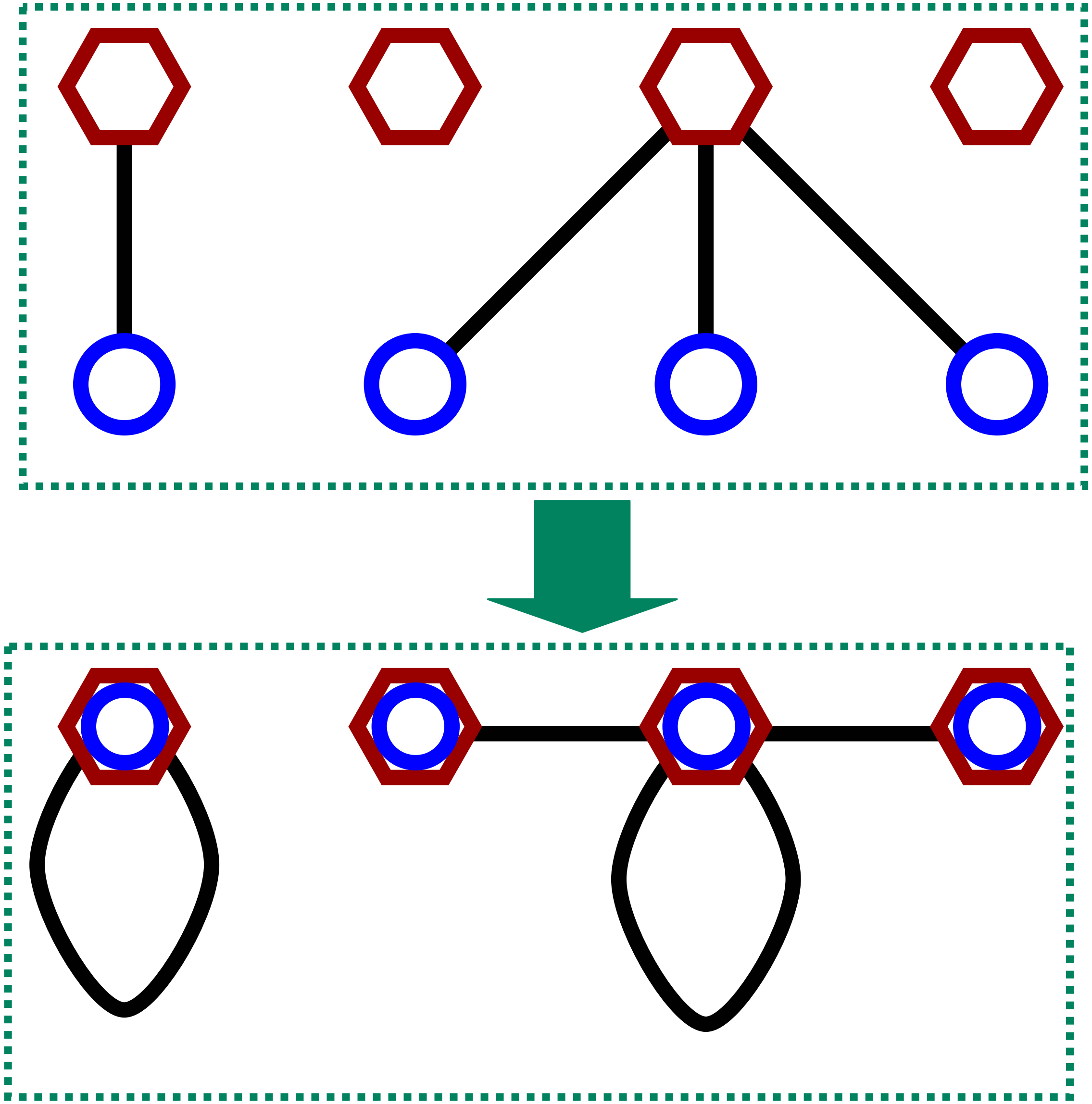
(E/2) edges

E individuals

This is just a **Molloy-Reed** [1995] projection onto a unipartite random graph of artifact vertices, with degree distribution p(k)

Equivalence to other network rewiring models (2)

- Alternative Projection 2:
  (N=E) Merge each individual vertex with one artifact vertex and let edges point from the individual to the artifact end. [Park et al. 2005]

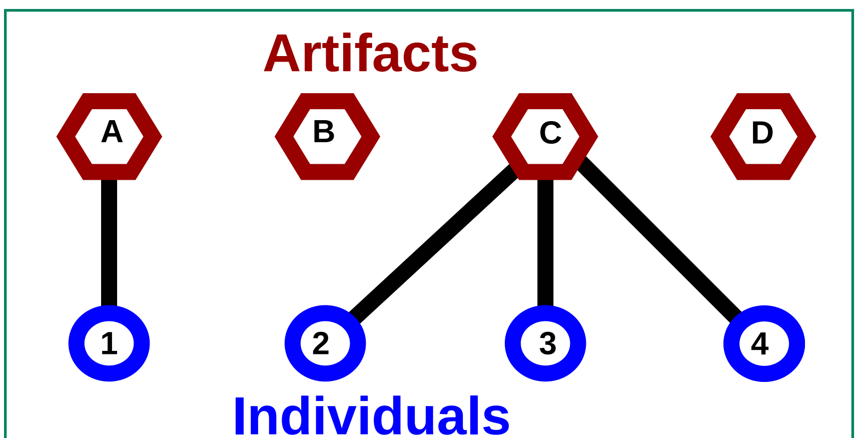


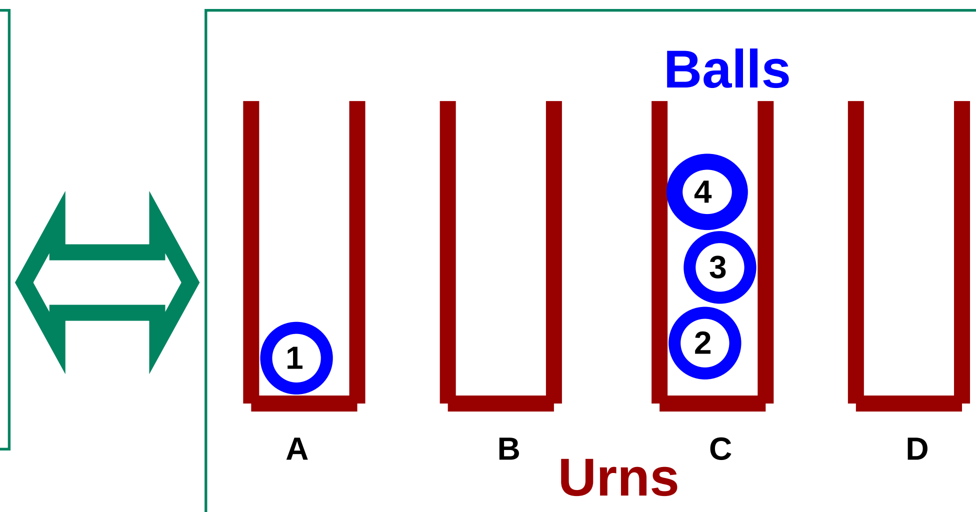
Relationship to Statistical Physics Models

Some parameter values of other models are equivalent to our model:

- Urn Models [Bernoulli 1713, …, Ohkubo et al. 2005]
- Zero Range Processes (Misanthrope version) [review M.R.Evans & Hanney 2005]
- Voter Models [Liggett 1999, …, Sood & Redner 2005]
- Backgammon/Balls-in-Boxes

applied to glasses [Ritort 1995], wealth distributions, simplicical gravity …





Relationship to Other Systems

- Gene Frequencies [Kimura and Crow, 1964]

  -Inheritence and Mutation

  Organisms (=individuals) ***inherit*** a copy of a gene (alleles = artifacts) leading to ***drift*** in genetic frequencies.
  Alternatively they gain a new ***mutation*** (random choice).

- Family Names [Zanette and Manrubia, 2001]

  -Inheritence and New Immigrants

  Males (=individuals) ***inherit*** family name (=artifacts).
  Occasionally new names appear ***randomly***
  (e.g. immigration).

- Language Extinction

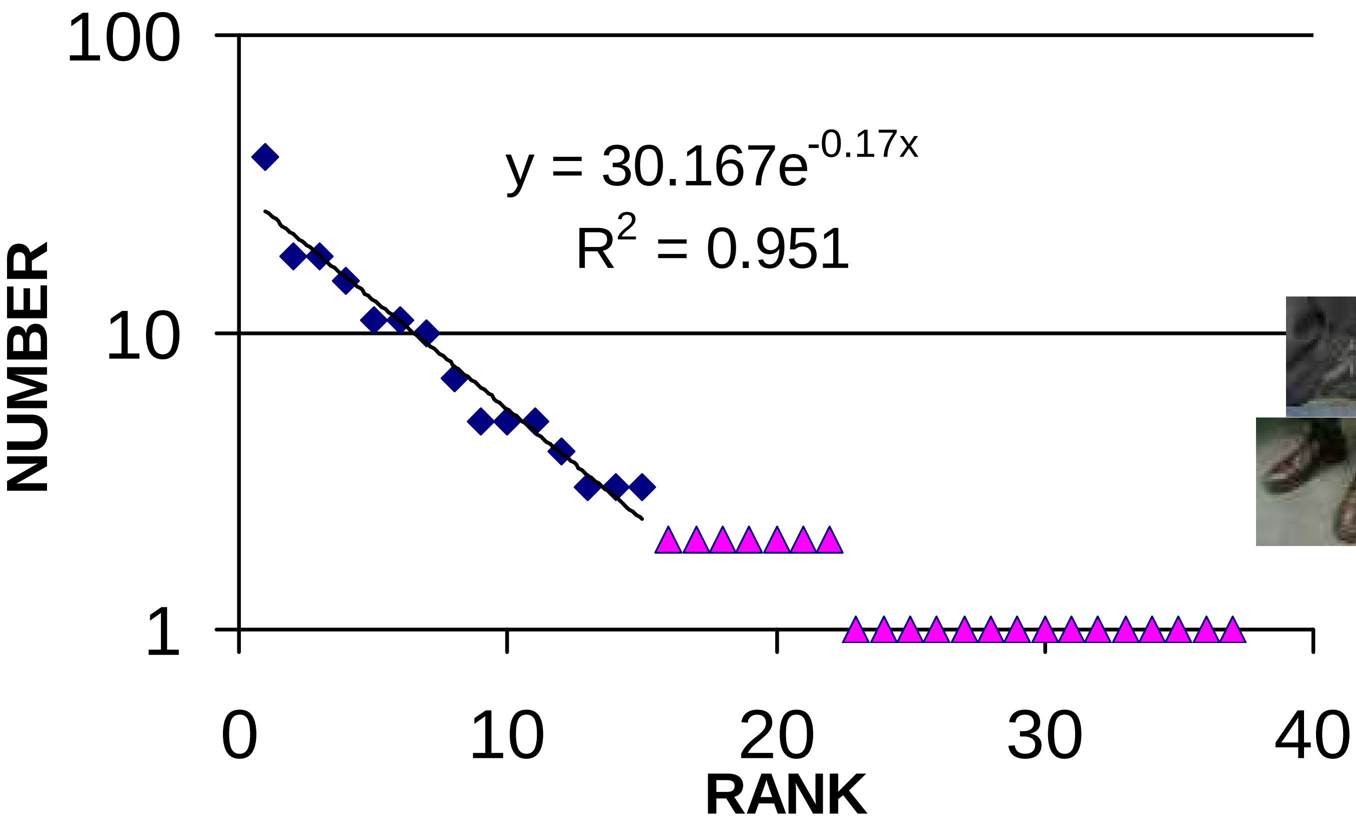
- Minority Game variant (see later)[Anghel et al, 2004]

Relationship to Other Systems

- Cultural Transmission [Bentley et al.,1999…2006]
  Individuals ***copy*** (p_p) the choice of artifact made by others or ***innovate*** (p_r)
  e.g. choice of pedigree dog,
  baby names,
  pop chart positions,
  archaeological pottery types,
  tennis star celebration action (?!),
  language extinction [Stauffer et al. 2006],
  fashion …

Relationship to Other Systems

- Cultural Transmission
  - Fashion (?) in the shoes of male physics students [Morgan and Swanell 2006]



186 Individuals in 196 categories of which 37 used, most popular white lace up trainers (39)

Mean Field Degree Distribution Master Equation

Mean field approximation very accurate for many models (low vertex correlations)

$$n(k,t+1) - n(k,t) = + n(k+1,t)\Pi_R(k+1)[1-\Pi_A(k+1)]$$
$$- n(k,t)\Pi_A(k)[1-\Pi_A(k)]$$
$$- n(k,t)\Pi_R(k)[1-\Pi_R(k)]$$
$$+ n(k-1,t)\Pi_A(k-1)[1-\Pi_R(k-1)]$$

(1-Π) terms Invariably ignored

Number of edges attaching to a vertex of degree (k-1)

Probability of NOT reattaching to same vertex

Can the Mean Field equation be exact?

Distribution $n_i(k)$ different in each instance i

Ensembles over many instances i

$$\left\langle \frac{n_i(k)k^{\beta}}{\sum_k n_i(k)k^{\beta}} \right\rangle \neq \left\langle n_i(k)k^{\beta} \right\rangle \left\langle \frac{1}{\sum_k n_i(k)k^{\beta}} \right\rangle$$

Normalisation of probabilities not usually same for different i

YES only if

$$\sum_k n_i(k)k^{\beta} = \left\langle \sum_k n_i(k)k^{\beta} \right\rangle$$

$\beta = 0$ or $\beta = 1$

Only Exactly Solvable Case

To be able to solve exactly we limit the attachment and removal probabilities, Π_R and Π_A, to be **linear** in degree exploiting only two constants of the motion, *N* and *E*

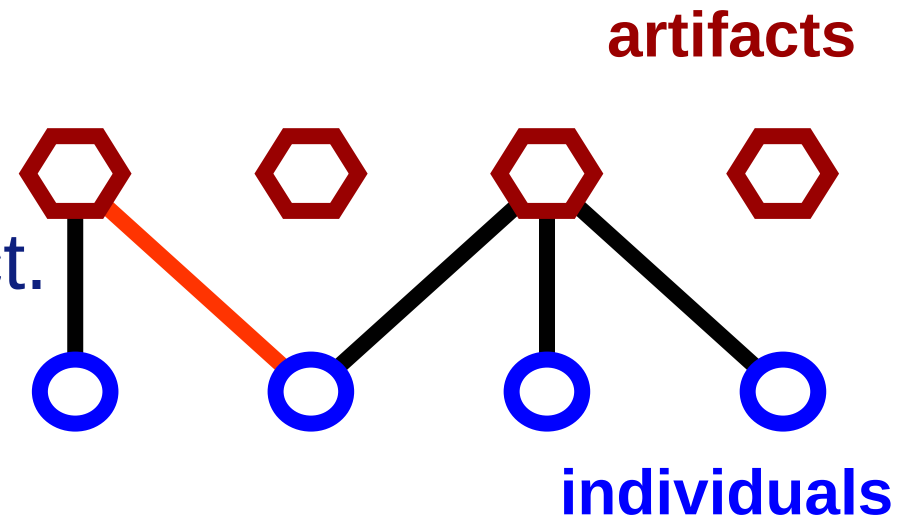
- $\Pi_R(k) = (k / E)$ Choose random edge to be rewired
- $\Pi_A(k) = [(1-p_r)k + p_r \langle k \rangle] / E$

Fraction $(1-p_r)$ of the time use preferential attachment

Fraction p_r of the time choose random attachment

Exact Mean Field rewiring processes

- **Removal**:
  A random individual decides to update their choice of artifact
- **Attachment**:
  With probability $(1-p_r)$ the individual **copies** the existing choice of *any* individual.
  With probability (p_r) the individual **innovates** by choosing a random artifact.



Exact Equilibrium Solution

$$n(k) = A \frac{\Gamma(k+\overline{K})}{\Gamma(k+1)} \frac{\Gamma(E-\overline{E}-\overline{K}-k)}{\Gamma(E+1-k)}$$

$$\overline{K} = \frac{p_r}{p_p}\langle k \rangle$$

$$\overline{E} = \frac{p_r}{p_p} E$$

A is ratio of four Γ functions

- Simple ratios of Γ functions
- Similar to those found for growing networks but second fraction is only found for network rewiring
- Only approximate solutions known previously

Large Degree Equilibrium Behaviour – Large p_r Case

For $p_r > p_* \sim 1/E$

(on average at least one edge attached to a randomly chosen artifact per generation)

$$\lim_{k\to\infty}\left[n(k)\right] = k^{-\gamma}\exp(-\zeta k)$$

$$\gamma = 1 - \frac{p_r}{p_p}\langle k\rangle$$

Power below one but in data indistinguishable from one

$$\zeta = -\ln(1 - p_r)$$

Exponential Cutoff

Large Degree Equilibrium Behaviour – Small p_r Case

For $p_r < p_* \sim 1/E$

(on average if all edges have been rewired once no edge is attached to a randomly chosen artifact per generation)

Degree distribution rises near $k=E$

$\Rightarrow$In extreme case $p_r=0$ all the edges are attached to ONE artifact

- a CONDENSATION or FIXATION

Blows up

$$n(k) = A\left(\frac{\Gamma(k+\overline{K})}{\Gamma(k+1)}\right)\left[\frac{\Gamma(E-\overline{E}-\overline{K}-k)}{\Gamma(E+1-k)}\right]$$

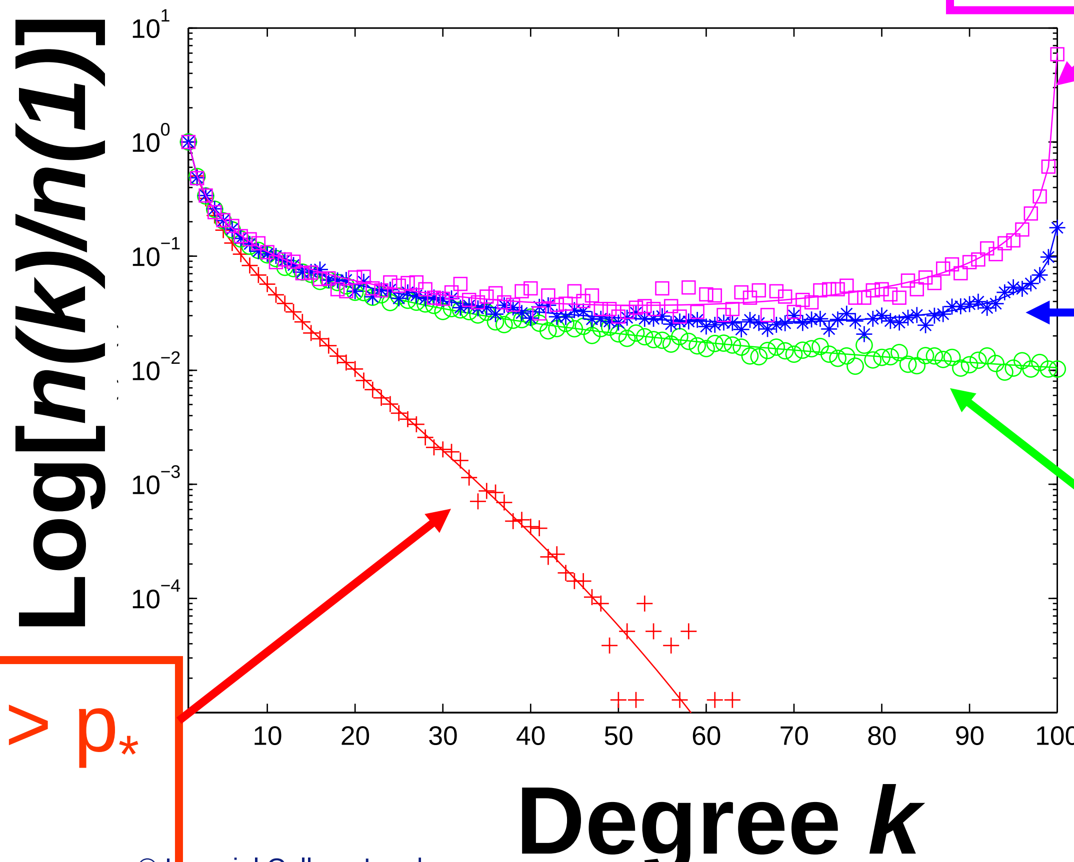
Equilibrium Behaviour Results

N=E=100

Points: 10^5 data runs

Lines: exact mean field solution

$p_r = 0.001 < p_*$
condensate



$p_r = 0.005$
$< p_*$

$p_r = 0.01$
$\cong p_*$
Almost pure
Power law

$p_r = 0.1 > p_*$
$\zeta^{-1} \cong 10$

Solution

Best solved using the generating function

$$\mathrm{G}(z,t) = \sum_{k=0}^{E} (z)^k \, \mathrm{n}(k,t) = \sum_{m=0}^{E} c_m (\lambda_m)^t \, \mathrm{G}^{(m)}(z)$$

where:-

- **Eigenfunctions** $\mathrm{G}^{(m)}(z) = (1-z)^m \, \mathrm{F}(a+m, b+m; c; z)$

Hypergeometric function

$$a = \frac{p_r}{p_p}\langle k \rangle, \quad b = -E, \quad c = 1 + a + b - \frac{p_r}{p_p}E$$

- **Eigenvalues** $\lambda_m = 1 - m(m-1)\dfrac{p_p}{E^2} - m\dfrac{p_r}{E}$

- **c_m are constants fixed by initial conditions**

Features of solution

- *n-th* moment of degree distribution gets contribuitions from only $m \leq n$ eigenfunctions
- $m=0$ eigenfunction number zero
  - only time independent solution = equilibrium
  - fixes distribution N
- $m=1$ eigenfunction *never contributes* otherwise would make first moment E time dependent
- Slowest time dependence comes from $m=2$ eigenfunction setting time scale

$$\tau_2 = -1/\ln(\lambda_2) \approx [2(p_r/E) + 2(1-p_r)/E^2]^{-1}$$

Homogeneity Measures F_n

- *n*-th derivatives of generating function gives measures of homogeneity related to *n*-th moment of degree distribution

$$F_n(t) := \frac{\Gamma(E+1-n)}{\Gamma(E+1)} \frac{d^n G(z,t)}{dz^n}\bigg|_{z=1} = \sum_{k=0}^{E} \frac{k}{E} \frac{(k-1)}{(E-1)} \cdots \frac{(k-n+1)}{(E-n+1)} n(k)$$

- **These are simple known ratios of Γ functions**

- **Equals the probability of choosing *n* different individuals connected to the same artifact**

$\Rightarrow$ $F_n = 0$ if no artifact chosen more than once

$F_n = 1$ if all individuals attached to same artifact

F_2 Homogeneity Measure

$$F_2(t) := \frac{1}{E(E+1)} \left. \frac{d^2 G(z,t)}{dz^2} \right|_{z=1}$$

F_2 = probability that two different individuals have chosen the same artifact

$$F_2(t) = F_2(0) + (\lambda_2)^t \left(F_2(\infty) - F_2(0) \right)$$

Initial values fix $F_2(0)$

e.g. $F_2(0)=0$ if each individual starts attached to unique individual

3rd eigenfunction controls all time dependence

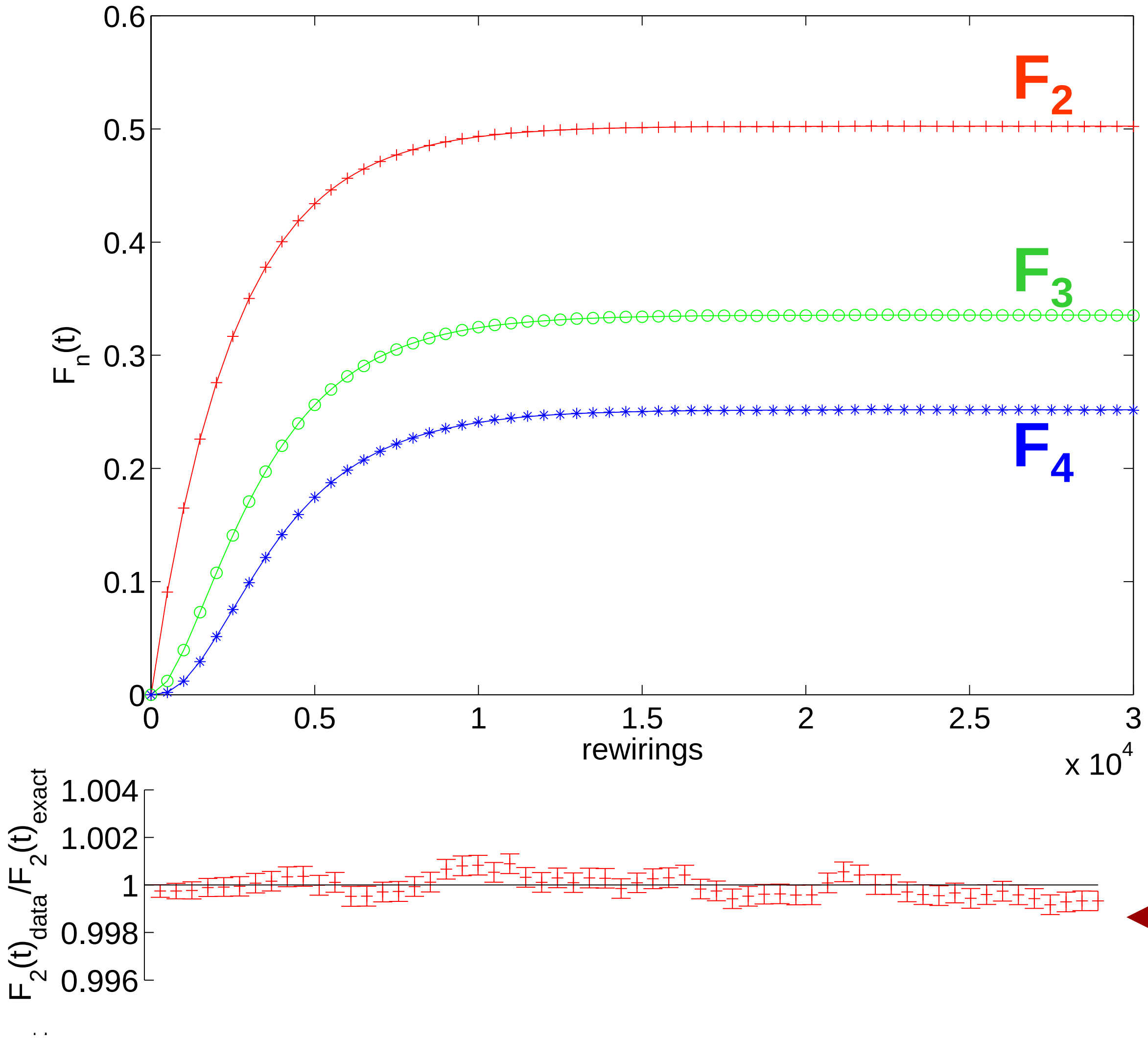
$\tau_2 = -1/\ln(\lambda_2)$

$\approx [2(p_r/E) + 2(1-p_r)/E^2]^{-1}$

$$F_2(\infty) = \frac{1 + p_r(\langle k \rangle - 1)}{1 + p_r(E-1)}$$

F_n numerical results

E=N=100, p_r=0.01≅p_*,
Points: average of 10^5 simulations
Lines: exact mean field prediction
Start: $n(k)=\delta_{k,1}$

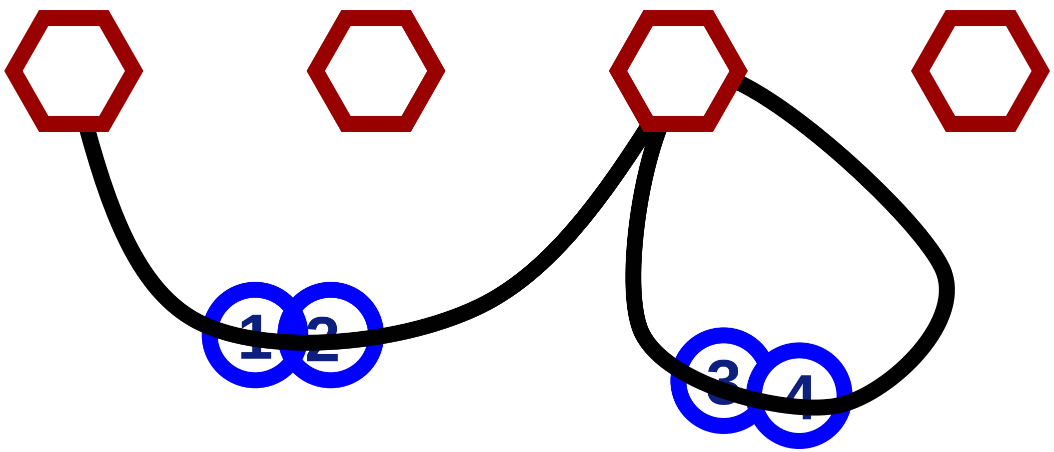


F increases as homogeneity increases with time

Time dependence of averages predicted very accurately, deviations less than 1%

Phase transitions in real time

- Bipartite graph can be projected onto a unipartite graph of the artifact vertices
- Artifact degree distribution p(k) is the degree distribution for a random graph



N artifacts

(E/2) edges

A Molloy-Reed [1995] projection

Graph Transition in Real Time

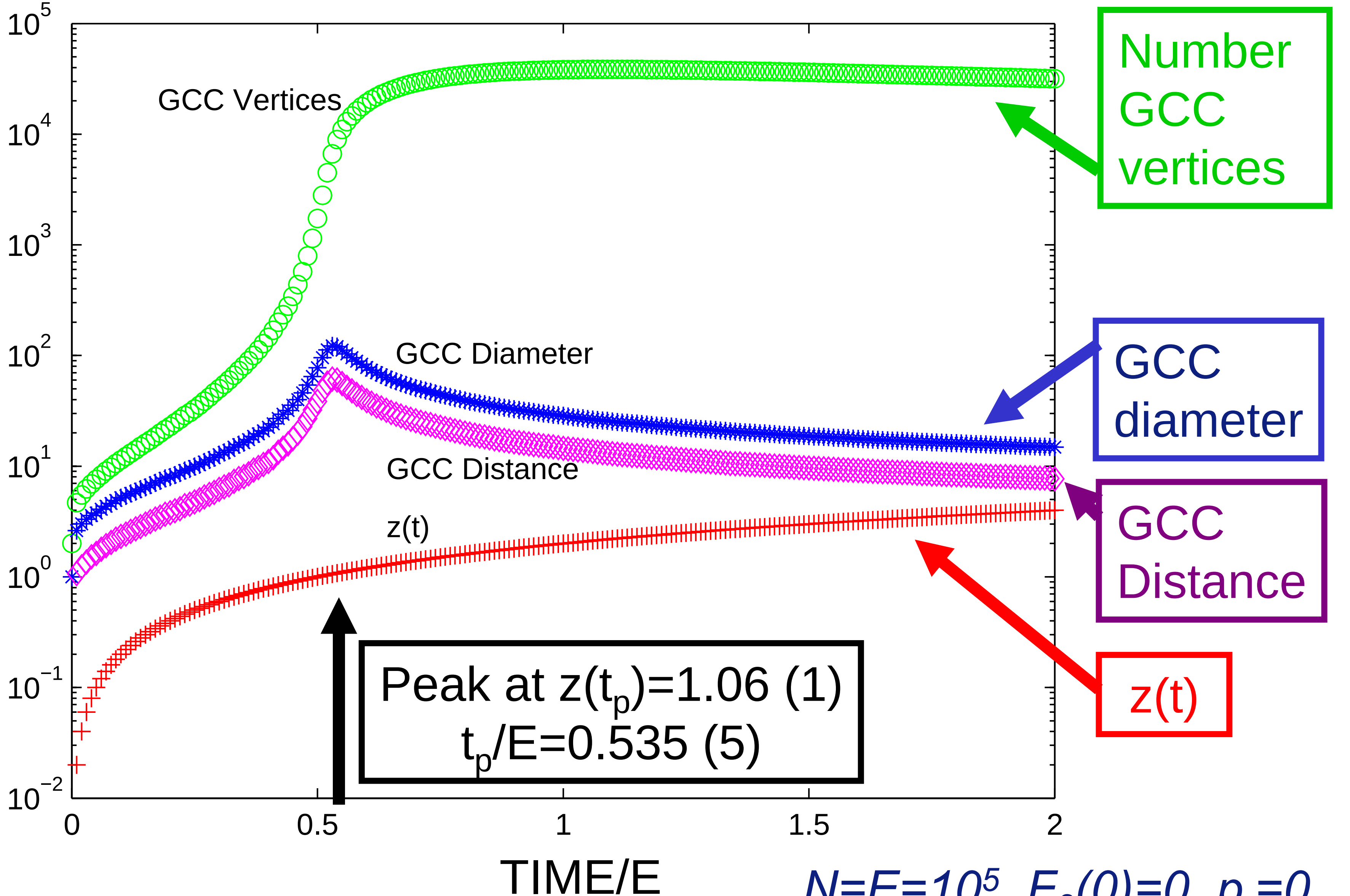
Infinite Random Graphs (given *p(k)* but otherwise completely randomised) have a phase transition (appearance of **GCC** - great connected component) at [Fronczak et al 2005, etc]

z(t)=1

where

$$z(t) = \frac{\langle k^2 \rangle}{\langle k \rangle} - 1 = (E-1)F_2(t)$$

Phase Transition in Molloy-Reed projection



$N=E=10^5,\ F_2(0)=0,\ p_r=0$

Phase Transition in Molloy-Reed projection

For $N=E=10^5$, $p_r=0$, initial $F_2(0)=0$

- $z(t)=1$ at $t=0.50000\ (2)$ as predicted
- Transition at $t/E = 0.535\ (5)$
- At transition $z(t)=1.06\ (1)$ not $z(t)=1$
- Average distance and diameter of GCC maximum at this point and second derivative of number of vertices in GCC zero at this point (within errors)

$\Rightarrow$**Finite size effects clearly present**

Generalisations of Model

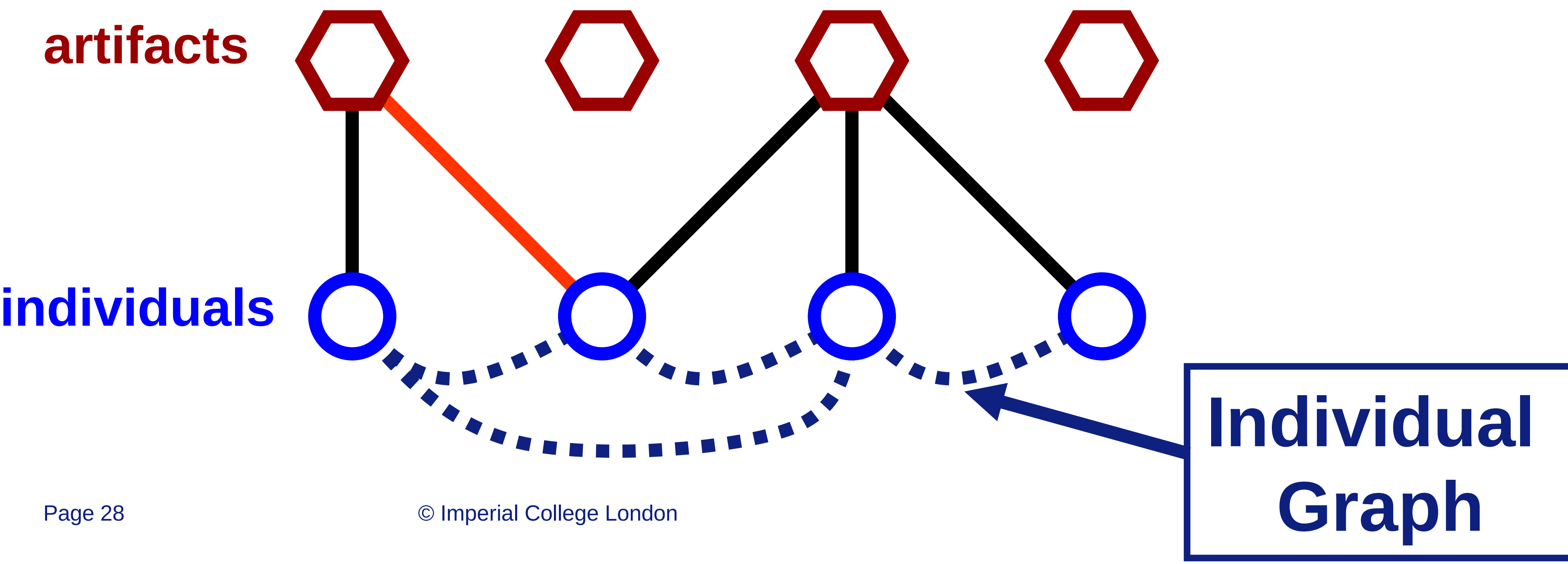
- Add a graph to the individual vertices
  -choose who to copy using individual's network

- Add a graph to the artifact vertices
  -mutations/innovations limited by metric in an artifact space

- Different types of individual
  -update their choice and copy/innovate at different rates

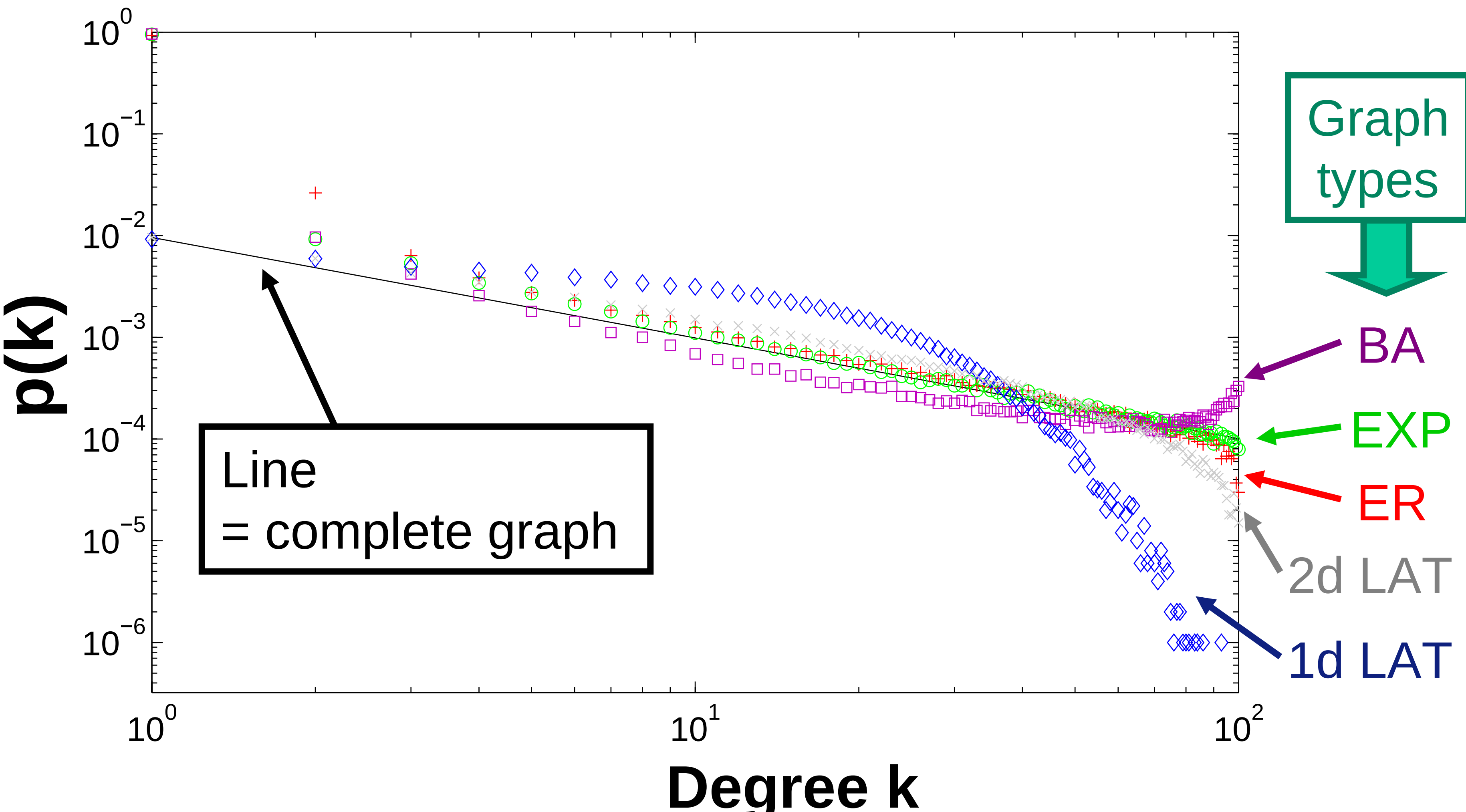
Adding a Network of Individuals

- **Removal**: Choose random individual as before
- **Attachment**:
  With probability *(1-p_r)* the individual **copies** the existing choice of any **neighbouring** individual.
  With probability *(p_r)* the individual **innovates**



Equilibrium with a Network of Individuals

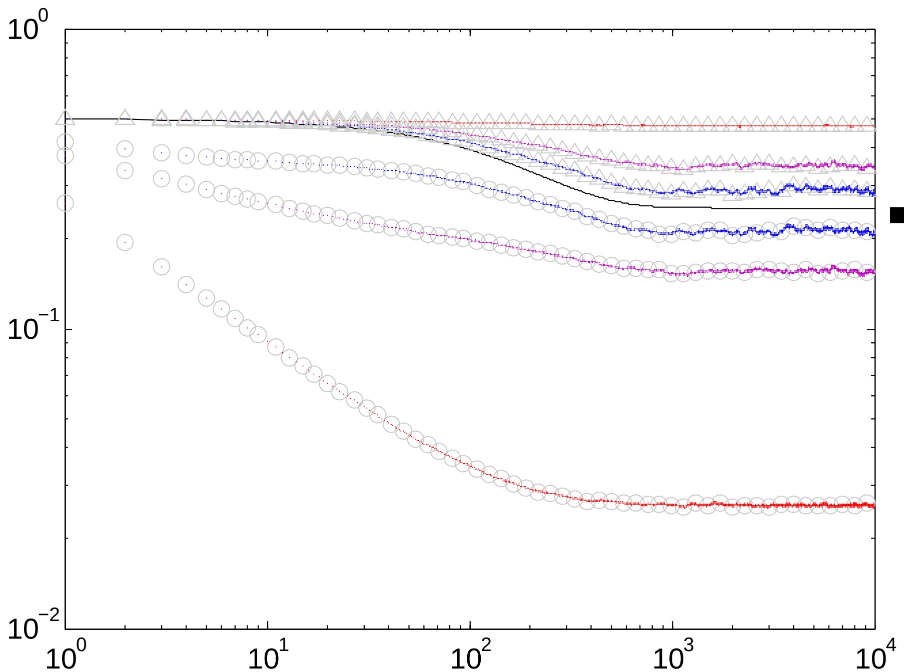
Qualitative behaviour largely unchanged except for 1d Lattice



N=E=100, p_r=1/E

Approach to Equilibrium for different Individual networks

- Results move away from complete graph as move from 3d -> 1d lattice



Global *(1-F_2)* gets bigger

Complete graph

Local homogeneity ρ gets smaller

TIME *N=2, p_r=1/E, E=729*

ρ = probability that n.n. has made different choice

Voter Model [Liggett 1999; Sood & Redner 2005]

- At each time step an individual is chosen randomly who copies the choice of a neighbour in an individual network
- Equivalent to $\boldsymbol{N=2, p_r=0}$ limit here
- Study time scales to come to complete **consensus** = condensation
- Used for models of language [Stauffer et al. 2006]

⇒We find approach to complete consensus is slow but a little randomness can speed this up while leaving a fairly complete condensation

Minority Game Example - Leaders and Followers

- At each step each individual chooses one or zero – the *minority* choice wins
- Choices are made based on one of a large but finite number of strategies using finite history – each strategy is a different artifact
- Individuals may follow their own prediction or they may follow the prediction from the most successful nearest neighbour in an ER random graph of individuals – i.e. they ***copy*** the strategy of a neighbour [Anghel et al. PRL 92 (2004) 058701]

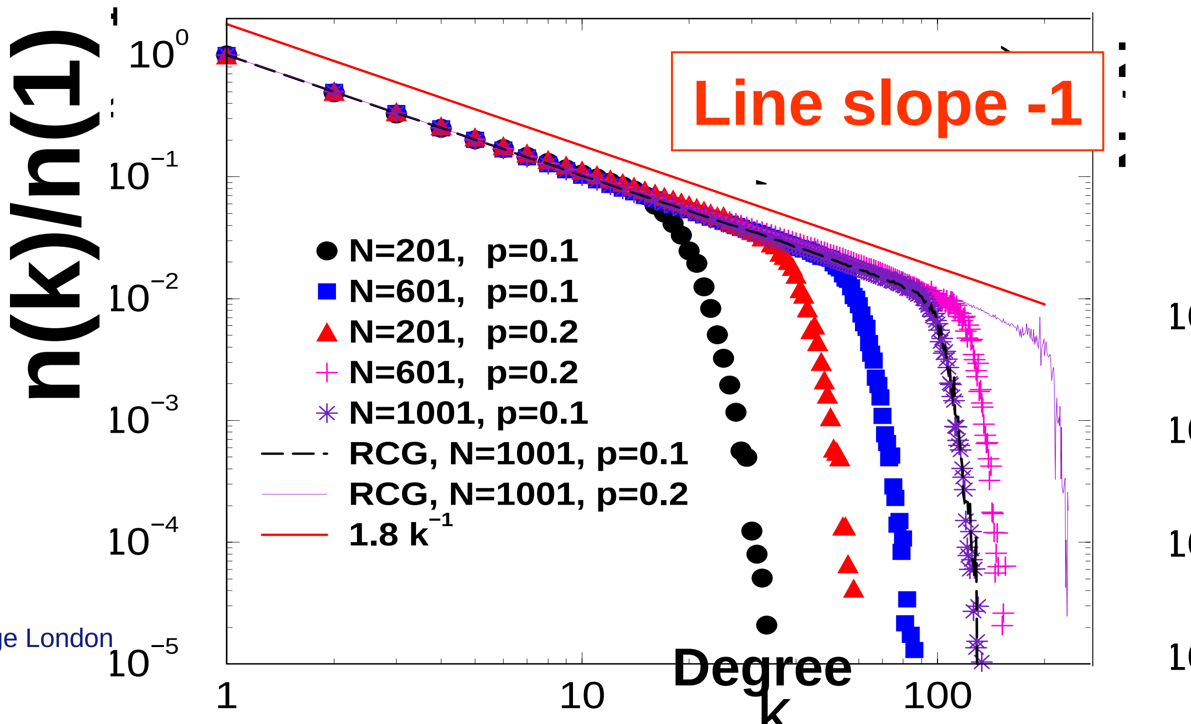
Minority Game Example – Leaders and Followers

Plot *n(k)* the average of the number of strategies (of some leader) used by *k* individuals (followers).

Various system sizes and various ER random graphs.

Result exactly as in our model

⇒ Random Copying



Minority Game Example - Leaders and Followers

Minority Game variant [Anghel et al, 2004]

- Agents (individual vertices) ***copy*** best strategy (artifacts) of their neighbours in an additional individual network.
- Number of people following a given strategy is effectively n(k) of our model.

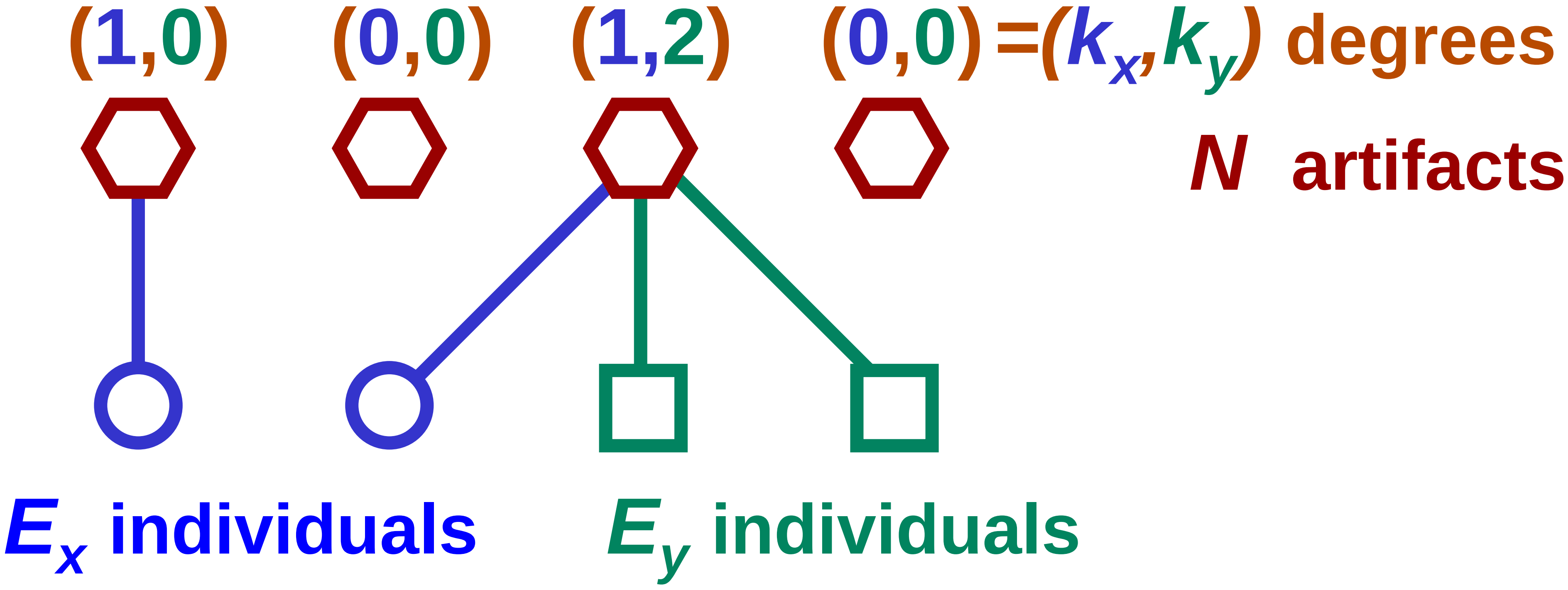
Shows how **copying** can arise naturally

- c.f. preferential attachment in growing networks

[TSE & Saramaki 2005]

Two Tribes

Change model so there are two types of individual, each type chooses new artifacts with their own probabilities for:- (A) copying from same type, (B) copying from different type, (C) innovation

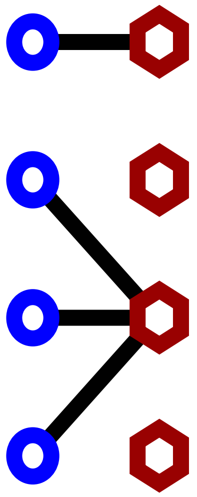


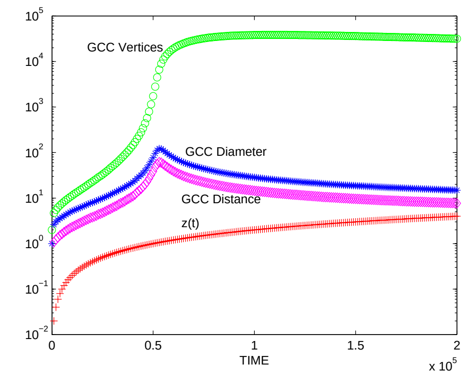
Two Tribes

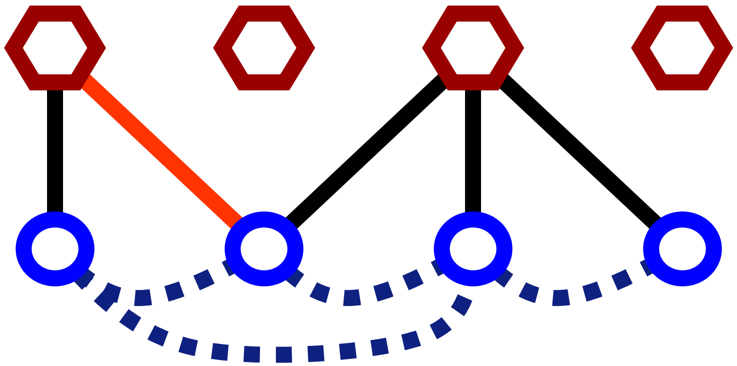
- Exact solutions for inhomogeneity measures $F_{2ab}(t)$ $[a,b \in \{X,Y\}]$ still possible
  - solutions of three-dimensional matrix
- 8 free parameters
  - difficult to draw general conclusions
- Might relate to *Freakonomics* type explanation for baby names in terms of different socioeconomic groups

Summary

- Made connections between rewiring of bipartite network and many other network, statistical physics and social science models.
  Some connections made in some existing papers.
- Exact mean field equation.
  Only now is behaviour at boundary k=E correct.
- Exact equilibrium solutions.
  Previous results for large degree k, large systems N,E.
- Exact solutions for all times in terms of standard functions – *phase transitions in time*
  I know of no other network solutions for arbitrary time and arbitrary size.







Summary

Many variations of model

- Individual Networks

  Only 1d lattice seems to make a big difference to equilibrium

- Generalisation of Voter models

  p_r can speed process up without significantly upsetting consensus

- Two Tribes

  exact solutions for some aspects possible with two types of individual